APPENDIX A
APPROXIMATED ALGORITHM

This section shows the pseudocode for the approximated algorithm used in Section VI. Such algorithm is adapted from [14] and is shown in Algorithm 7. Namely, given a network $N = (V, E, s, t, c)$, a budget $B \in \mathbb{N}$ and $\varepsilon > 0$, the following variables are maintained:

- $\mathcal{F}_{\text{out}}$ is an outer ILP to be solved for each iteration of the main loop in lines 6–14. Each iteration of such loop modifies $\mathcal{F}_{\text{out}}$ by adding two new constraints (lines 10 and 14). As for the decision variables, there are always $|E|$ binary variables x_{ij} and one real variable η.
- $\eta^*, \boldsymbol{x}^*$ is the current approximation for the *GMF* and the removed edges, respectively. They are possibly set in line 9 and returned in line 13.
- $\eta^{\text{out}}, \hat{\boldsymbol{x}}$ is the value for the *GMF* and the removed edges for the current iteration.
- $\hat{\eta}$ is the value for the maximum flow of the network as damaged by the current $\hat{\boldsymbol{x}}$.

We also use an auxiliary function *MaxFlow* which returns the maximum flow of an input network $\hat{\eta}$ and the flow f_{ij} for all $(i, j) \in E$ (see Definition III.2). Such function is called in line 8, where the edges selected by $\hat{x}$ have been removed from the input network. Finally, a further auxiliary function *Random* (I) is used to selected uniformly at random an element from a set I.

1 **function** *ApproxNIP*$(V, E, s, t, c, B, \varepsilon)$
2 $\quad \eta^* \leftarrow \infty$;
3 $\quad \hat{\boldsymbol{x}} \leftarrow \text{Random}(\{\boldsymbol{x} \in \{0,1\}^{|E|} \mid \sum_{(i,j)\in E} x_{ij} = B\})$;
4 $\quad \mathcal{F}_{\text{out}} \leftarrow \{\max_{\boldsymbol{x} \mid \sum_{(i,j)\in E} x_{ij} = B} \eta\}$;
5 $\quad \mathcal{F}_{\text{out}} \leftarrow \mathcal{F}_{\text{out}} \cup \{\forall (i,j) \in E.\ x_{ij} \in \{0,1\}\}$;
6 $\quad$ **while** true **do**
7 $\quad\quad \hat{E} \leftarrow \{(i,j) \in E \mid \hat{x}_{ij} = 0\}$;
8 $\quad\quad \hat{\eta}, \hat{\boldsymbol{f}} \leftarrow \text{MaxFlow}(V, \hat{E}, s, t, c)$;
9 $\quad\quad$ **if** $\hat{\eta} < \eta^*$ **then** $\eta^*, \boldsymbol{x}^* \leftarrow \hat{\eta}, \hat{\boldsymbol{x}}$;
10 $\quad\quad \mathcal{F}_{\text{out}} \leftarrow \mathcal{F}_{\text{out}} \cup \{\eta \leq \hat{\eta} + \sum_{(i,j)\in E} \hat{f}_{ij} x_{ij}\}$;
11 $\quad\quad \eta^{\text{out}}, \hat{\boldsymbol{x}} \leftarrow \text{ILPSolver}(\mathcal{F}_{\text{out}})$;
12 $\quad\quad$ **if** $\eta^{\text{out}} - \eta^* \leq \varepsilon \eta^*$ **then**
13 $\quad\quad\quad$ **return** $\eta^*, \boldsymbol{x}^*$;
14 $\quad\quad \mathcal{F}_{\text{out}} \leftarrow \mathcal{F}_{\text{out}} \cup \{\sum_{(i,j)\in E} \hat{x}_{ij} x_{ij} \leq B - 1\}$;

Algorithm 7: Approximated algorithm

APPENDIX B
PROOFS OF THEOREMS AND LEMMAS

In this section, we will provide the full proofs for theorems stated in Section V-B. To this aim, we first provide useful lemmas in Section B-A. Then, we show that our graph transformations are correct in Section B-B. Finally, Section B-C proves the theorems from Section V-B.

A. Useful Theorems and Lemmas

In the following, an s, t-cut in a network $N = (V, E, s, t, c)$ is a partition (S, T) of V (*i.e.*, $S \cap T = \emptyset$, $S \cup T = V$) s.t. $s \in S, t \in T$. The capacity of an s, t-cut is the sum of the capacities of the crossing edges, *i.e.*, $c(S, T) = \sum_{(u,v)\in(S\times T)\cap E} c(u, v)$.

We start by recalling an important theorem from graph theory, namely the min-cut-max-flow Theorem B.1 from [11].

Theorem B.1 (Max-Flow-Min-Cut). *Let $N = (V, E, s, t, c)$ be a network. The maximum flow of N is equal to the minimum capacity of an s, t-cut, i.e., MaxFlow$(N) = \min\{c(S, T) \mid (S, T)$ is an (s, t)-cut of $N\}$.*

We will also use the following lemma.

Lemma B.1 (NIP for subnetwork). *Let (N, B) be a Network Interdiction Problem (NIP), with $N = (V, E, s, t, c)$. Let $N' = (V', E', s, t, c')$ be a subnetwork of N having the same maximum flow of N. That is, $V' \subseteq V, E' \subseteq E, \forall (x, y) \in E'$ $c'(x, y) = c(x, y)$ and MaxFlow$(N) =$ MaxFlow(N').*

Then, $GMF(N, B) = GMF(N', B)$ for all budgets $B \in \mathbb{N}$. Furthermore, if α is an optimal solution for (N', B), then an optimal solution for (N, B) is $\beta(u, v) = \alpha(u, v)$ if $(u, v) \in E'$ and $\beta(u, v) = 0$ otherwise.

Proof. We prove the Lemma by induction on B. As for the induction basis, we have that $GMF(N, 0) =$ *MaxFlow*$(N) =$ *MaxFlow*$(N') = GMF(N', 0)$. As for the induction step, let us assume that $GMF(N, B) = GMF(N', B)$, and we want to prove that $GMF(N, B + 1) = GMF(N', B + 1)$. This is obviously true if $GMF(N, B) = GMF(N', B) = 0$ (as $GMF(N, B + 1) \leq GMF(N, B)$), thus we focus on $GMF(N, B) = GMF(N', B) > 0$. Before proving this, we note that, for any optimal attack β for $(N, B+1)$, there exists a maximum flow f on N such that $\beta(u, v) = 1 \rightarrow f(u, v) > 0$ (recall that $GMF(N, B) > 0$). In fact, if there exists an edge (u, v) s.t. $\beta(u, v) = 1$ and, for all maximum flows f, $f(u, v) = 0$, then a better attack β' can be defined by selecting an edge (w, z) s.t. $\beta(w, z) = 0 \wedge f(w, z) > 0$ (such an edge exists, otherwise $GMF(N, B) = 0$), *i.e.*, $\beta'(u, v) = 0, \beta'(w, z) = 1$ and $\beta'(x, y) = \beta(x, y)$ for all other $(x, y) \notin \{(u, v), (w, z)\}$.

Assume now by absurd that $GMF(N, B + 1) > GMF(N', B+1)$, and let α, α' be optimal attacks on N, N' respectively. The same attack α', if applied on N', has the same value *MaxFlow*$(N|_{\alpha'}) =$ *MaxFlow*$(N'|_{\alpha'})$, as edges and costs of N' are the same in N. Thus, α is not optimal, a contradiction.

If instead $GMF(N, B+1) < GMF(N', B+1)$, let α, α' be optimal attacks on N, N' respectively. This is only possible if $\alpha(u, v) = 1$ for some $(u, v) \notin E'$, otherwise α has the same value *MaxFlow*$(N|_\alpha) =$ *MaxFlow*$(N'|_\alpha)$ and α' is not optimal for N'. However, by the considerations above, $\alpha(u, v) = 1 \rightarrow f(u, v) > 0$, and being the maximum flow the same in N and N', we have that $(u, v) \in E'$, which is contradiction.

The last part of the lemma follows from the discussion above.

□

B. Correctness of Transformations

In this section we prove that our transformations are correct, *i.e.*, i) they preserve the *GMF* of the original network, and ii) an attack for the original network can be computed from an attack on the transformed network.

We start by proving that the reachability reduction is correct.

Lemma B.2 (Reachability reduction correctness)**.** *Let (N, B) be a NIP, with $N = (V, E, s, t, c)$. Let $N' = (V', E', s, t, c')$ be the network defined as follows:*

- $V' = \left\{ v \in V \;\middle|\; \begin{array}{l} \exists u_1, \ldots, u_k \\ \quad u_1 = s \wedge u_k = t \wedge \\ \quad \exists i \in [1, k] \;\; u_i = v \wedge \\ \quad \forall i \in [1, k-1] \;\; (u_i, u_{i+1}) \in E \end{array} \right\}$;
- $E' = \{(u, v) \in E \mid u, v \in V' \wedge s \neq v \wedge t \neq u\}$;
- $c'(u, v) = c(u, v)$ *for* $(u, v) \in E'$.

Then, $GMF(N, B) = GMF(N', B)$*. Furthermore, if α is an optimal solution for (N', B), then an optimal solution for (N, B) is $\beta(u, v) = \alpha(u, v)$ if $(u, v) \in E'$ and $\beta(u, v) = 0$ otherwise.*

Proof. By Lemma B.1, as N' is a subnetwork of N, we only have to prove that *MaxFlow*(N) = *MaxFlow*(N'). Consider an s, t-cut S, T of N having the minimum value $c(S, T)$ = *MaxFlow*(N). Let $(u, v) \in E$ be a crossing edge (*i.e.*, $u \in S, v \in T$), then u is reachable from s. In fact, if u is not reachable from s, then we can define a new s, t-cut $\tilde{S} = S \cap F, \tilde{T} = V \setminus \tilde{S} = T \cup F$, being F the set of nodes which are backward reachable from u. By construction, (u, v) is not a crossing edge any more and the new crossing edges are a proper subset of the old ones (*i.e.*, $(\tilde{S} \times \tilde{T}) \cap E \subset (S \times T) \cap E$). Thus, $c(\tilde{S}, \tilde{T}) < c(S, T)$, *i.e.*, (S, T) is not minimal. Using a dual argument, it can be shown that crossing edges cannot contain a node which is not backward reachable from t, *i.e.*, $(u, v) \in (S \times T) \cap E$ implies that there exists a path from v to t. Hence, we have that $c(S, T) = c'(S \cap V', T \cap V')$ and *MaxFlow*(N) = *MaxFlow*(N').

□

We now prove that chain compaction is correct.

Lemma B.3 (Chain compaction correctness)**.** *Let (N', B) be a NIP, with $N' = (V', E', s, t, c')$, and let $\Gamma = \{\langle v_1 \ldots, v_r \rangle \mid r \geq 3 \wedge \forall i \in [1, r-1] \; (v_i, v_{i+1}) \in E' \wedge \forall i \in [2, r-1] \; \mathrm{indegree}(v_i) = \mathrm{outdegree}(v_i) = 1 \wedge \forall i \in \{1, r\} \; \mathrm{indegree}(v_i) \neq 1 \vee \mathrm{outdegree}(v_i) \neq 1\}$ be the set of maximal chains in N'. Furthermore, for all (u, v) s.t. $(u, v) \in E'' \wedge (u, v) \notin E'$, let $\Gamma^{-1}(u, v) = \gamma \in \Gamma(N')$ s.t. $v = \gamma_{|\gamma|}$ and either $u = \gamma_1$ or $u = \gamma_2$. Let $N'' = (V'', E'', s, t, c'')$ be the network defined as follows:*

- $E'' = ((E' \setminus E_1) \setminus E_2) \cup E_3 \cup E_4$, *being:*
  - $E_1 = \{(u, v) \mid \exists \gamma \in \Gamma, i \geq 2 : u = \gamma_i, v = \gamma_{i+1}\}$;
  - $E_2 = \{(u, v) \mid \exists \gamma \in \Gamma \wedge (\gamma_1, \gamma_{|\gamma|}) \notin E' \wedge u = \gamma_1, v = \gamma_2\}$;
  - $E_3 = \{(u, v) \mid \exists \gamma \in \Gamma \wedge (\gamma_1, \gamma_{|\gamma|}) \notin E' \wedge u = \gamma_1, v = \gamma_{|\gamma|}\}$;
  - $E_4 = \{(u, v) \mid \exists \gamma \in \Gamma \wedge (\gamma_1, \gamma_{|\gamma|}) \in E' \wedge u = \gamma_2, v = \gamma_{|\gamma|}\}$;
- $V'' = \{v \mid \exists (u, v) \in E'' \vee \exists (v, u) \in E''\}$;
- $c''(u, v) = m(\Gamma^{-1}(u, v))$ *for all* $(u, v) \in E'' \setminus E'$*, and* $c''(u, v) = c'(u, v)$ *for all other edges* $(u, v) \in E'' \cap E'$.

Then $GMF(N', B) = GMF(N'', B)$*. Furthermore, if α is an optimal solution for (N'', B), then an optimal solution for (N', B) is β defined as: for all $(u, v) \in E' \cap E''$, $\beta(u, v) = \alpha(u, v)$, for all $(x, y) \in E'' \setminus E'$ s.t. $\alpha(x, y) = 1$, $\beta(\gamma_1, \gamma_2) = 1$ being $\gamma = \Gamma^{-1}(x, y)$, and $\beta(u, v) = 0$ for all other $(u, v) \in E'$.*

Proof. Since N'' is not a subnetwork of N' (for each chain in N', a new edge is also added to N''), we cannot directly apply Lemma B.1. Thus, we prove the lemma by induction on B. As for the induction basis, we have to prove that $GMF(N', 0)$ = *MaxFlow*(N') = *MaxFlow*(N'') = $GMF(N'', 0)$. Consider an s, t-cut S, T of N' having the minimum value $c'(S, T)$ = *MaxFlow*(N'). Let $\gamma \in \Gamma$ be a chain in N', which implies its nodes are reachable from s and can reach t, and let $E'(\gamma) = \{(u, v) \mid u, v \in \gamma\}$ be the set of edges in γ. Then, either γ does not cross the cut S, T or γ crosses the cut S, T with an edge having minimum cost $m(\gamma)$, in formulas $\forall i \; (\gamma_i, \gamma_{i+1}) \notin (S \times T)$ or $\exists! i \; (\gamma_i, \gamma_{i+1}) \in (S \times T) \wedge c'(\gamma_i, \gamma_{i+1}) = m(\gamma)$. In fact, if the crossing edge is not the one with minimum cost, we could easily build a new cut with a smaller value by having an edge with minimum cost as the lone crossing edge. In formulas, if $\gamma_i \in S, \gamma_{i+1} \in T$ and $c'(\gamma_i, \gamma_{i+1}) > m(\gamma)$, with $m(\gamma) = c'(\gamma_j, \gamma_{j+1})$ for some j, then the new s, t-cut would be $\tilde{S}, \tilde{T}$ defined as $\tilde{S} = S \setminus \gamma \cup \{v \in V' \mid \exists k \leq j \; v = \gamma_k\}, \tilde{T} = V' \setminus \tilde{S}$. By construction, $c'(S, T) > c'(\tilde{S}, \tilde{T})$, a contradiction. With a similar reasoning, we can rule out that a chain has more than one crossing edge, *i.e.*, *card*$((S \times T) \cap E' \cap E'(\gamma)) > 1$. Hence, we have that $c'(S, T) = c''(S \cap V'', T \cap V'')$ and *MaxFlow*(N') = *MaxFlow*(N'').

We now assume that $GMF(N', B) = GMF(N'', B) > 0$, and we want to prove that $GMF(N', B + 1) = GMF(N'', B+1)$. Let α be an optimal attack for $(N', B+1)$, we define the set of chains $\tilde{\Gamma} \subseteq \Gamma(N')$ s.t. α contains one edge with minimum cost from each chain in $\tilde{\Gamma}$, in formulas $\tilde{\Gamma} = \{\gamma \in \Gamma \mid \exists i \; \alpha(\gamma_i, \gamma_{i+1}) = 1\}$. Note that, from the discussion above, for all $\gamma \in \tilde{\Gamma}$, there exists only one edge (γ_i, γ_{i+1}) s.t. $\alpha(\gamma_i, \gamma_{i+1}) = 1$, otherwise α is not optimal. We will refer to such edge (γ_i, γ_{i+1}) as γ_α. Furthermore, for $\gamma \in \Gamma$, let $E''(\gamma)$ be the edge added to E' in order to handle chain γ, *i.e.*, $E''(\gamma) = (\gamma_1, \gamma_{|\gamma|})$ if $(\gamma_1, \gamma_{|\gamma|}) \notin E'$ and $E''(\gamma) = (\gamma_2, \gamma_{|\gamma|})$ otherwise (note that $\Gamma^{-1}(E''(\gamma)) = \gamma$). By construction, for all chains $\gamma \in \tilde{\Gamma}$, $c'(\gamma_\alpha) = c''(E''(\gamma)) = m(\gamma)$.

Now, let β be defined from α by replacing all attacked chains $\gamma \in \tilde{\Gamma}$ with the corresponding added edge $E''(\gamma)$. In formulas, for all $\gamma \in \tilde{\Gamma}$, $\beta(E''(\gamma)) = 1$, while for all $\gamma \in \Gamma \setminus \tilde{\Gamma}$, $\beta(E''(\gamma)) = 0$, and $\beta(u, v) = \alpha(u, v)$ for all other $(u, v) \in E''$. By construction, β is also optimal for $(N'', B + 1)$, thus we have the thesis $GMF(N, B + 1) = GMF(N', B + 1)$.

The last part of the lemma follows from the discussion above.

□

Furthermore, also detour reduction is correct.

Lemma B.4 (Detour elimination correctness)**.** *Let (N'', B) be a NIP, with $N'' = (V'', E'', s, t, c'')$. Let $N''' = (V''', E''', s, t, c''')$ be the network defined as follows:*

- $V''' = \left\{ u \in V'' \;\middle|\; \begin{array}{l} \exists u_1, \ldots, u_k \;\; u_1 = s \wedge u_k = t \wedge \\ \forall i \in [1, k-1] \;\; (u_i, u_{i+1}) \in E'' \wedge \\ (\forall i, j \in [1, k] \;\; i \neq j \rightarrow u_i \neq u_j) \wedge \\ \exists i \in [1, k] \;\; u_i = u \end{array} \right\}$;

- $E''' = \{(u,v) \in E'' \mid u,v \in V'''\}$;
- $c'''(u,v) = c''(u,v)$ *for all* $(u,v) \in E'''$.

Then $GMF(N'',B) = GMF(N''',B)$. *Furthermore, if* α *is an optimal solution for* (N''',B)*, then an optimal solution for* (N'',B) *is* $\beta(u,v) = \alpha(u,v)$ *if* $(u,v) \in E''$ *and* $\beta(u,v) = 0$ *otherwise.*

Proof. By Lemma B.1, as N''' is a subnetwork of N'', we only have to prove that *MaxFlow*(N'') = *MaxFlow*(N'''). Consider an s,t-cut S,T of N' having the minimum value $c(S,T)$ = *MaxFlow*(N''). Let $(u,v) \in E$ be a crossing edge (*i.e.*, $u \in S, v \in T$), then (u,v) is not a detour edge, *i.e.*, $(u,v) \in E'''$. In fact, if $(u,v) \in E'' \setminus E'''$ is a detour edge, the we can define a new s,t-cut $\tilde{S},\tilde{T}$ by moving v to $\tilde{S}$, together with all detour nodes that are reachable from v and backward reachable from u. In formulas, $\tilde{S} = S \cup$

$$\left\{ w \in V''' \setminus V'' \;\middle|\; \begin{array}{l} \exists u_1,\ldots,u_k \\ \quad u_1 = s \wedge u_k = w \,\wedge \\ \quad \exists j \in [1,k] \;\; u_j = u \,\wedge \\ \quad \forall i \in [1,k-1] \;\; (u_i,u_{i+1}) \in E'' \end{array} \right\}$$

and $\tilde{T} = V \setminus \tilde{S}$.

By construction, (u,v) is not a crossing edge any more and the new crossing edges are a proper subset of the old ones (*i.e.*, $(\tilde{S} \times \tilde{T}) \cap E'' \subset (S \times T) \cap E''$). Thus, $c''(\tilde{S},\tilde{T}) < c''(S,T)$, *i.e.*, (S,T) is not minimal. Hence, we have that $c''(S,T) = c'''(S \cap V''', T \cap V''')$ and *MaxFlow*(N'') = *MaxFlow*(N'''). □

Lemma B.4 may be relaxed as described in Corollary B.1, which is used in the proof of Theorem V.1 in Section V-B.

Corollary B.1 (Relaxation of detour elimination correctness)**.** *Let* (N'',B) *be a NIP, with* $N'' = (V'',E'',s,t,c'')$ *and let* $N''' = (V''',E''',s,t,c''')$ *be obtained as in Lemma B.4. Let* $D \subseteq V'' \setminus V'''$ *be a set of detour nodes and let* $M = (W,F,s,t,d)$ *be the network defined as follows:*

- $W = V'' \setminus D$;
- $F = \{(u,v) \in E'' \mid u,v \in W\}$;
- $d(u,v) = c''(u,v)$ *for all* $(u,v) \in F$.

Then $GMF(N'',B) = GMF(M,B)$. *Furthermore, if* α *is an optimal solution for* (M,B)*, then an optimal solution for* (N'',B) *is* $\beta(u,v) = \alpha(u,v)$ *if* $(u,v) \in E''$ *and* $\beta(u,v) = 0$ *otherwise.*

Proof. The proof follows the same lines of Lemma B.4. □

Finally, by combining the previous lemmas, we may state that using our transformations in sequence is correct.

Lemma B.5. *Let* (N,B) *be an instance of a NIP, with* $N = (V,E,s,t,c)$. *Let* (N',B) *be obtained as described in the hypothesis of Lemma B.2,* (N'',B) *be obtained, starting from* (N',B)*, as described in the hypothesis of Lemma B.3 and* (N''',B) *be obtained, starting from* (N'',B)*, as described in the hypothesis of Lemma B.4.*

Then, $GMF(N,B) = GMF(N''',B)$. *Furthermore, if* α *is an optimal solution for* $(N''',B) = (V''',E''',s,t,c''')$*, then an optimal solution for* (N,B) *is* β *defined as: for all* $(u,v) \in E \cap E'''$*,* $\beta(u,v) = \alpha(u,v)$*, for all* $(x,y) \in E''' \setminus E$ *s.t.* $\alpha(x,y) = 1$*,* $\beta(\gamma_1,\gamma_2) = 1$ *being* $\gamma = \Gamma^{-1}(x,y)$*, and* $\beta(u,v) = 0$ *for all other* $(u,v) \in E$.

Proof. This is a corollary of Lemma B.2, Lemma B.3 and Lemma B.4. □

Analogously to Corollary B.1, we may state Corollary B.2 as a corollary of Lemma B.5.

Corollary B.2. *Let* (N,B) *be an instance of a NIP, with* $N = (V,E,s,t,c)$. *Let* (N',B) *be obtained as described in the hypothesis of Lemma B.2,* (N'',B) *be obtained, starting from* (N',B)*, as described in the hypothesis of Lemma B.3 and* (M,B) *be obtained, starting from* (N'',B)*, as described in the hypothesis of Corollary B.1.*

Then, $GMF(N,B) = GMF(M,B)$. *Furthermore, if* α *is an optimal solution for* $(M,B) = (W,F,s,t,d)$*, then an optimal solution for* (N,B) *is* β *defined as: for all* $(u,v) \in E \cap F$*,* $\beta(u,v) = \alpha(u,v)$*, for all* $(x,y) \in F \setminus E$ *s.t.* $\alpha(x,y) = 1$*,* $\beta(\gamma_1,\gamma_2) = 1$ *being* $\gamma = \Gamma^{-1}(x,y)$*, and* $\beta(u,v) = 0$ *for all other* $(u,v) \in E$.

Proof. This is a corollary of Lemma B.2, Lemma B.3 and Corollary B.1. □

C. Correctness and Complexity of Algorithms

We now prove Theorem V.1 from Section V-B.

Theorem B.2. *Let* (N,B) *be a NIP instance, with* $N = (V,E,s,t,c)$. *Furthermore, let* $((\tilde{V},\tilde{E},s,t,\tilde{c}),\eta,\beta)$ *be the result of calling function* ReduceAndSolveNIP *with arguments* N,B.

Then, the following holds:

1) $\eta = GMF(N,B)$.
2) β *is an optimal solution for* (N,B).

Proof. As for point 1), we start noting that V'',E'',c'' of Lemma B.3 are the same as $\hat{V},\hat{E},\hat{c}$ computed in line 4 of Algorithm 2. In fact, function *ReachChainReduct* perform the reachability reduction as described in Lemma B.2, while at the same time compacting the chains as described in Lemma B.3.

As for detours, we apply Corollary B.1 to the set D computed in line 2 of Algorithm 5, which is a subset of the detour nodes, and considering the discussion above, we have that point 1) holds by Lemma B.5.

As for point 2), it follows from 1) and from Lemma B.5, since the computation of β in lines 7–12 of Algorithm 2 follows the lines of Lemma B.5. □

Finally, we prove Theorem V.2 from Section V-B.

Theorem B.3. *Let* (N,B) *be a NIP instance, with* $N = (V,E,s,t,c)$. *The worst-time complexity of function* ReduceAndSolveNIP *with arguments* N,B *is* $O(|V|^2(|V| + |E|))$.

Proof. The worst-case scenario occurs when, for an input network $N = (V,E,s,t,c)$, no reduction occurs, *i.e.*, $\tilde{V} = \hat{V} = V$ and $\tilde{E} = \hat{E} = E$. In such a case, for function *ReachChainReduct*, the complexity is $O(|V| + |E|)$, as it is obtained from a standard Depth-First Search (DFS). As for function *DetourReduct*, the worst-case scenario occurs when (V,E) is a complete graph (*i.e.*, for all nodes $u,v \in V$ there exists an edge $(u, v) \in E$). Given this, since for each reachable node in V (lines 11–17 in Algorithm 5), $|V|$ nested DFS could be started (lines 15–16 in Algorithm 5), we have that the worst-time complexity is $O(|V|^2 (|V| + |E|))$. $\square$